

NOTE ON THE WEIERSTRASS PREPARATION THEOREM IN QUASIANALYTIC LOCAL RINGS

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ABSTRACT. Consider quasianalytic local rings of germs of smooth functions closed under composition, implicit equation, and monomial division. We show that if the Weierstrass Preparation Theorem holds in such a ring then all elements of it are germs of analytic functions.

1. INTRODUCTION AND MAIN RESULTS

Since the original work of Borel [4, 5], the notion of quasianalytic rings of infinitely differentiable functions has been studied intensively (see for example the expository article on quasianalytic local rings written by V. Thilliez [16]). Recall that a ring \mathcal{C}_n of smooth germs at the origin of \mathbb{R}^n is called *quasianalytic* if the only element of \mathcal{C}_n which admits a zero Taylor expansion is the zero germ.

The early works of Denjoy [8] and Carleman [6] show a deep connection between the growth of partial derivatives of \mathcal{C}^∞ germs at the origin and the quasianalyticity property, leading to the notion of *quasianalytic Denjoy-Carleman classes of functions*. The algebraic properties of such rings, namely their stability under several classical operations, such as composition, differentiation, implicit function, is well understood (see for example [13] and [11]).

These stability properties have allowed a study of quasianalytic classes from the point of view of *real analytic geometry*, that is the investigation of the properties of subsets of the real spaces locally defined by equalities and inequalities satisfied by elements of these rings. For example, it is shown in [2] how the resolution of singularities extends to the quasianalytic framework.

. However, two classical properties, namely Weierstrass division and Weierstrass preparation, seem to cause trouble in the quasianalytic setting. For example it has been proved by Childress in [7] that quasianalytic Denjoy-Carleman classes might not satisfy Weierstrass division. Since Weierstrass preparation is usually introduced as a consequence of Weierstrass division, it is classically considered that Weierstrass preparation should fail in a quasianalytic framework. So far, no explicit counterexample has been given. Moreover, we do not know any example of a ring of smooth functions for which the Weierstrass preparation theorem holds, but the Weierstrass division fails.

. We are interested here in what we call in the next section a *quasianalytic system*, that is a collection of quasianalytic rings of germs of smooth functions which contain the analytic germs, and which is closed under composition, partial differentiation and implicit function. Such systems have been investigated in several works from the point of view of real analytic geometry or o-minimality [9, 15, 2, 14]. It is

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worth noticing that, in these papers, the possible failure of Weierstrass preparation leads to a study mostly based on resolution of singularities.

. In such a context, a nice result has been obtained by Elkhadiri and Soufli in [10]. They prove, in a remarkably simple way, that if a quasianalytic system satisfies Weierstrass division, then it coincides with the analytic system: all its germs are analytic. The proof is based on the following idea. In order to prove that a given real germ f is analytic at the origin of \mathbb{R}^n , they prove that f extends to a holomorphic germ at the origin of \mathbb{C}^n . This extension is built by considering the *complex formal extension* $\hat{f}(x + iy) \in \mathbb{C}[[x, y]]$, where \hat{f} is the Taylor expansion of f at the origin. The real and imaginary parts of this series satisfy the Cauchy-Riemann equations. Moreover, the Weierstrass division of $f(x + t)$ by the polynomial $t^2 + y$ shows that these real and imaginary parts are the Taylor expansions of two germs which belong to the initial quasianalytic system. By quasianalyticity, these two germs also satisfy the Cauchy-Riemann equations. They consequently provide the real and imaginary parts of a holomorphic extension of f .

. Our goal is to use Elkhadiri and Soufli's methods to prove that a quasianalytic system in which Weierstrass preparation holds coincides with the analytic system. This result apply in particular to the examples of quasianalytic systems mentioned above. We still don't know if any of the rings they contain (besides the rings of germs in one variable) is noetherian or not.

. A similar property of failure of Weierstrass preparation has been announced in [1] for the quasianalytic Denjoy-Carleman classes. The approach there, pretty different than ours, leads to a precise investigation of the following *extension problem*: does a function belonging to a quasianalytic Denjoy-Carleman class defined on the positive real axis extends to a function belonging to a wider Denjoy-Carleman class defined on the real axis? The authors actually produce an explicit example of non-extendable function with additional properties which permit contradicting Weierstrass preparation.

2. NOTATIONS AND MAIN RESULT.

Notation. For $n \in \mathbb{N}$, we denote by \mathcal{E}_n the ring of smooth germs at the origin of \mathbb{R}^n and by $\mathcal{A}_n \subset \mathcal{E}_n$ the subring of analytic germs.

For every $f \in \mathcal{E}_n$, we denote by $\hat{f} \in \mathbb{R}[[x_1, \dots, x_n]]$ its (infinite) Taylor expansion at the origin.

Finally, we denote (x_1, \dots, x_n) by \mathbf{x} and (x_1, \dots, x_{n-1}) by \mathbf{x}' .

Definition 2.1. Consider a collection $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ of rings of germs of smooth functions at the origin of \mathbb{R}^n . We say that \mathcal{C} is a *quasianalytic system* if the following properties hold for all $n \in \mathbb{N}$:

- (1) The algebra \mathcal{A}_n is contained in \mathcal{C}_n .
- (2) (Stability by composition) If $f \in \mathcal{C}_n$ and $g_1, \dots, g_n \in \mathcal{C}_m$ with $g_1(0) = \dots = g_n(0) = 0$, then $f(g_1, \dots, g_n) \in \mathcal{C}_m$.
- (3) (Stability by implicit equation, assuming $n > 0$) If $f \in \mathcal{C}_n$ satisfies $f(0) = 0$ and $(\partial f / \partial x_n)(0) \neq 0$, then there exists $\varphi \in \mathcal{C}_{n-1}$ such that $\varphi(0) = 0$ and $f(\mathbf{x}', \varphi(\mathbf{x}')) = 0$.
- (4) (Stability under monomial division) If $f \in \mathcal{C}_n$ satisfies $f(\mathbf{x}', 0) = 0$, then there exists $g \in \mathcal{C}_n$ such that $f(\mathbf{x}) = x_n g(\mathbf{x})$.

- (5) (Quasianalyticity) For every $n \in \mathbb{N}$, the Taylor map $f \mapsto \hat{f}$ is injective on \mathcal{C}_n .

Remark 2.2. It can easily be seen that the above properties imply that the algebras \mathcal{C}_n are closed under partial differentiation (see [14, p.423] for example).

Definition 2.3. A germ $f \in \mathcal{E}_n$ is of order n in the variable x_n if $f(\mathbf{0}, x_n) = x_n^d u(x_n)$, where $u(0) \neq 0$ (that is, u is a unit of \mathcal{E}_1).

Definition 2.4. We say that a quasianalytic system \mathcal{C} satisfies *Weierstrass preparation* if, for all $n \in \mathbb{N}$, the following statement (\mathcal{W}_n) holds : every $f \in \mathcal{C}_n$ of order d in the variable x_n can be written

$$f = U(\mathbf{x}) (x_n^d + a_1(\mathbf{x}') x_n^{d-1} + \cdots + a_d(\mathbf{x}')) ,$$

where $U \in \mathcal{C}_n$, $a_1, \dots, a_d \in \mathcal{C}_{n-1}$, $U(0) \neq 0$ and $a_1(0) = \cdots = a_d(0) = 0$.

Our main result is the following :

Theorem. *If the quasianalytic system \mathcal{C} satisfies Weierstrass Preparation, then it coincides with the analytic system: for all $n \in \mathbb{N}$, $\mathcal{C}_n = \mathcal{A}_n$.*

Remark 2.5. We will actually prove that the conclusion of the theorem is true once \mathcal{W}_3 holds.

3. PROOF OF THE THEOREM

We consider in this section a quasianalytic system \mathcal{C} which satisfies Weierstrass Preparation.

. In order to prove the theorem, it is enough to prove that $\mathcal{C}_1 = \mathcal{A}_1$. In fact, it is noticed in [10] that the equality $\mathcal{C}_1 = \mathcal{A}_1$ implies $\mathcal{C}_n = \mathcal{A}_n$ for all $n \in \mathbb{N}$. The argument is the following. If $f \in \mathcal{C}_n$ (and $n > 1$) then, for every $\xi \in \mathbb{S}^{n-1}$, the germ $f_\xi : t \mapsto f(t\xi)$ belongs to \mathcal{C}_1 . Hence, under the assumption $\mathcal{C}_1 = \mathcal{A}_1$, the germ f_ξ is analytic. Thanks to a result of [3], this implies that $f \in \mathcal{A}_n$.

Lemma. *Let $f \in \mathcal{C}_n$ such that $f(\mathbf{0}, x_n) = x_n^2 + x_n^3 + h(x_n)$, where $h \in \mathcal{C}_1$ has order greater than 3. Then there exists $f_0, f_1 \in \mathcal{C}_n$ such that*

$$f(\mathbf{x}) = f_0(\mathbf{x}', x_n^2) + x_n f_1(\mathbf{x}', x_n^2) .$$

Proof. We introduce the germs $g_0 : \mathbf{x} \mapsto (f(\mathbf{x}', x_n) + f(\mathbf{x}', -x_n))/2$ and $g_1 : \mathbf{x} \mapsto (f(\mathbf{x}', x_n) - f(\mathbf{x}', -x_n))/2$, which both belong to \mathcal{A}_n and satisfy $f = g_0 + g_1$. They are respectively even and odd in the variable x_n . Hence the exponents of x_n in their Taylor expansions at the origin are respectively even and odd.

The order of g_0 in the variable x_n is exactly 2, so is the order in x_n of the germ $F : (\mathbf{x}, t) \mapsto g_0(\mathbf{x}) - t$, which belongs to \mathcal{C}_{n+1} . Since the system \mathcal{C} satisfies Weierstrass preparation, there exist $\varphi_1, \varphi_2 \in \mathcal{C}_n$ and a unit $U \in \mathcal{C}_{n+1}$ such that $\varphi_1(\mathbf{0}) = \varphi_2(\mathbf{0}) = 0$ and

$$F(\mathbf{x}, t) = (x_n^2 + \varphi_1(\mathbf{x}', t)x_n + \varphi_2(\mathbf{x}', t)) \cdot U(\mathbf{x}, t) .$$

We claim that $\varphi_1 = 0$. In fact, considering the Taylor expansions, we have:

$$\hat{F}(\mathbf{x}, t) = (x_n^2 + \hat{\varphi}_1(\mathbf{x}', t)x_n + \hat{\varphi}_2(\mathbf{x}', t)) \cdot \hat{U}(\mathbf{x}, t) .$$

Now it stems from the classical proof of Weierstrass preparation theorem for formal series that the support of $x_n^2 + \hat{\varphi}_1(\mathbf{x}', t)x_n + \hat{\varphi}_2(\mathbf{x}', t)$ is contained in the sub-semigroup of \mathbb{N}^{n+1} generated by the support of \hat{F} . Hence this support contains only

even powers of the variable x_n , and $\hat{\varphi}_1 = 0$. Since the system \mathcal{C} is quasianalytic (point 5 of Definition 2.1), $\varphi_1 = 0$.

Notice that the order of the germ $(\mathbf{x}', z, t) \mapsto z + \varphi_0(\mathbf{x}', t)$ in the variable t is 1. Since the system \mathcal{C} is closed under implicit equation (point 3 of Definition 2.1), there exists a germ $f_0 \in \mathcal{C}_n$ such that

$$z + \varphi_0(\mathbf{x}', t) = 0 \iff t = f_0(\mathbf{x}', z).$$

We deduce that

$$\begin{aligned} t = g_0(\mathbf{x}) &\iff F(\mathbf{x}, t) = 0 \iff x_n^2 + \varphi_0(\mathbf{x}', t) = 0 \\ &\iff t = f_0(\mathbf{x}', x_n^2), \end{aligned}$$

that is $g_0(\mathbf{x}) = f_0(\mathbf{x}', x_n^2)$.

In the same way, we notice that the order of g_1 in the variable x_n is exactly 3. Moreover, $g_1(\mathbf{x}', 0) = 0$. By stability under monomial division (point 4 of Definition 2.1) there exists $\bar{g}_1 \in \mathcal{C}_n$ such that $g_1(\mathbf{x}) = x_n \bar{g}_1(\mathbf{x})$. The germ \bar{g}_1 is even in the variable x_n and its order in this variable is exactly 2.

Therefore there exists a germ $f_1 \in \mathcal{C}_n$ such that $\bar{g}_1(\mathbf{x}) = f_1(\mathbf{x}', x_n^2)$, and the lemma is proved. \square

Remark. Given a germ $f \in \mathcal{E}_n$, the existence of f_0 and f_1 in \mathcal{E}_n which satisfy the statement of the lemma is a well known fact (see for example [12, p. 12]). But the classical proof, whose first step consists in transforming f in a flat germ, cannot work in a quasianalytic system.

Proof of the Theorem. Consider a germ $h \in \mathcal{C}_1$. Up to adding a polynomial, we may suppose that $h(x_1) = x_1^2 + x_1^3 + \ell(x_1)$, where the order of ℓ in the variable x_1 is greater than 3. We define the germ $f \in \mathcal{C}_2$ by $f: (x_1, x_2) \mapsto h(x_1 + x_2)$. According to the lemma, there exist two germs f_0 and f_1 in \mathcal{C}_2 such that

$$f: (x_1, x_2) \mapsto f_0(x_1, x_2^2) + x_2 f_1(x_1, x_2^2).$$

We introduce the complex germ H defined by

$$H: z = x_1 + ix_2 \in \mathbb{C} \mapsto f_0(x_1, -x_2^2) + ix_2 f_1(x_1, -x_2^2).$$

We see that $H(x_1, 0) = f(x_1, 0) = h(x_1)$. Hence the theorem is proved once we have proved that the germ H is holomorphic, that is that its real and imaginary parts satisfy the Cauchy-Riemann equations.

Consider the Taylor expansion $\hat{h}(x_1) = \sum_{n \geq 0} h_n x_1^n \in \mathbb{R}[[x_1]]$ of the germ h . The real and imaginary parts of the formal series $\hat{H}(x_1 + ix_2) \in \mathbb{C}[[x_1, x_2]]$ defined by

$$\hat{H}(x_1 + ix_2) = \sum_{n \geq 0} h_n (x_1 + ix_2)^n$$

are the series $\hat{f}_0(x_1, -x_2^2)$ and $x_2 \hat{f}_1(x_1, -x_2^2)$. Since the coefficients h_n , $n \in \mathbb{N}$, are real numbers, these series satisfy the Cauchy-Riemann equations. By quasianalyticity, the germs $f_0(x_1, -x_2^2)$ and $x_2 f_1(x_1, -x_2^2)$ satisfy the same equations.

We deduce that the complex germ H is holomorphic, and thus the germ h is analytic. \square

Remark 3.1. In the proof of the theorem, the lemma is applied to the germ f which belongs to \mathcal{C}_2 . Hence the single hypothesis \mathcal{W}_3 is actually required.

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